

A Simple Proof of the BPH Theorem¹

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Abstract

A new formalism is given for the renormalization of quantum field theories to all orders of perturbation theory, in which there are manifestly no overlapping divergences. We prove the BPH theorem in this formalism, and show how the local subtractions add up to counterterms in the action. Applications include the renormalization of lattice perturbation theory, the decoupling theorem, Zimmermann oversubtraction, the renormalization of operator insertions, and the operator product expansion.

1 Introduction

The first approach to showing that the divergences of quantum field theories may be absorbed into local counterterms to all orders in perturbation theory was due to Dyson [1, 2]. The R-operation was introduced by Bogoliubov and Parasiuk [3, 4] following some earlier work of Stückelberg and Green [5], and their proof that this removed all the divergences was corrected by Hepp [6].

Our approach is based on several ideas. The idea of differentiation with respect to external momenta comes from Tarasov and Vladimirov [7, 8] and Chetyrkin, Kataev, and Tkachev [9]. The approach to proving the equivalence of the subtractions made by the R-operation and counterterms is due to Anikin, Polivanov, and Zavialov [10]. These ideas were combined to provide a proof of the BPH theorem by Caswell and Kennedy [11], which however did not provide a completely satisfactory proof that subtracted integrals which were overall (power-counting) convergent were actually convergent. The idea (but not the name) of a “small momentum cutoff” was introduced by Hahn and Zimmermann [12], and the Henge decomposition was introduced by Caswell and Kennedy [13] and applied to the closely related problem of studying the asymptotic large-momentum behaviour of convergent Feynman diagrams.

The present work combines and extends these methods to give a proof of the BPH theorem which makes no use of Feynman parameters. This is important as the proof is applicable to lattice perturbation theory where the propagators are not quadratic forms in the momenta, and the usual Feynman parameterization is not applicable [14, 15]. The present proof is only directly applicable to theories in Euclidean space: for lattice field theory this is all that is needed, and for theories with quadratic propagators the corresponding Minkowski space results follow from the Euclidean space ones [16]. We also require that there are no massless propagators in order to avoid infrared divergences.

2 Graphs and Integrals

We need to make a few elementary definitions if for no other reason than to specify our notation. A graph is *connected* if it cannot be partitioned into two sets of vertices which are not connected by an edge. A graph is *one particle irreducible* (1PI) if it remains connected after removing any edge. A single vertex is thus a 1PI graph. A *Feynman integral* $I(\mathcal{G})$ may be associated with any graph \mathcal{G} by means of the *Feynman rules* for the theory. A propagator is associated with each line, some factor with each vertex, and a D -dimensional momentum integral with each independent closed loop. $I(\mathcal{G})$ is a function of the external momenta p , the lightest mass m (we assume $m > 0$ to avoid infrared divergences), some dimensionless couplings, and a cutoff $\Lambda \equiv 1/a$ which is introduced to make the theory well defined. We extend the mapping $I : \mathcal{G} \mapsto I(\mathcal{G})$ to act linearly on sums of graphs.

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2.1 Derivatives of Graphs and Taylor's Theorem

It is useful to consider the *derivative* of a Feynman diagram with respect to its external momenta. This is drawn diagrammatically as $\partial \text{---}\bigcirc\text{---} = \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}$. Note that we view crossed lines and vertices as associated with new Feynman rules: although one might view the cross as a new vertex inserted into a line this notation is not adequate in general when vertices (including such crosses themselves) have a non-trivial momentum dependence.


Each of the graphs shown above is really a sum over all the components of all the independent external momenta, $I(\partial\mathcal{G}) = \frac{\partial I(\mathcal{G})}{\partial p_\mu}$, $I(\partial^2\mathcal{G}) = \frac{\partial^2 I(\mathcal{G})}{\partial p_\mu \partial p_\nu}$, and so forth. Viewing $I(\mathcal{G})$ as a function of its external momenta, repeated application of the fundamental theorem of calculus gives us Taylor's theorem. In our notation $I(p) = T^n I(p) + \int_{p_0}^p dp_1 \int_{p_0}^{p_1} dp_2 \dots \int_{p_0}^{p_{n-1}} dp_n \partial^n I(p_n)$, where

$$T^n I(p) \equiv \sum_{j=0}^n \frac{(p-p_0)^j}{j!} \partial^j I(p_0) = \sum_{j=0}^n \sum_{\mu_1, \dots, \mu_j} \frac{(p-p_0)_{\mu_1} \dots (p-p_0)_{\mu_j}}{j!} \frac{\partial^j I(p_0)}{\partial p_{\mu_1} \dots \partial p_{\mu_j}}.$$

2.2 Henges

Any graph may be decomposed into a set of disjoint 1PI components and a set of edges which do not belong to any 1PI subgraph. Selecting any line from a graph defines a *henge*, which is just the set of 1PI components of the graph with the specified line removed. An example of a henge is , where the heavy

lines indicate the set of 1PI subgraphs in the henge corresponding to the light line. We shall write \mathcal{G}/\mathcal{H} to indicate the graph obtained by shrinking each 1PI subgraph Θ in \mathcal{H} to a point. If \mathcal{G} is a 1PI graph and $\ell \in \mathcal{G}$ some edge, then \mathcal{G} may be considered as a single loop $\mathcal{G}/\mathcal{H}(\mathcal{G}, \ell)$ with the 1PI subgraphs in the

henge $\mathcal{H}(\mathcal{G}, \ell)$ acting as “effective vertices.” For the example above the graph \mathcal{G}/\mathcal{H} is . The set of all henges for a four-loop contribution to the two-point function of ϕ^3 theory is $\left\{ \text{---}\bigcirc\text{---}, \text{---}\bigcirc\text{---}, \text{---}\bigcirc\text{---}, \text{---}\bigcirc\text{---}, \text{---}\bigcirc\text{---} \right\}$;

the henges $\mathcal{H}(\mathcal{G}, \ell)$ shown as heavy lines correspond to ℓ being any of the light lines.

We define $I_\lambda(\mathcal{G})$ to be the Feynman integral corresponding to \mathcal{G} where all the lines carry momentum greater than λ ; that is $|k_\ell| > \lambda$ ($\forall \ell \in \mathcal{G}$) where we use the usual Euclidean norm. This corresponds to Feynman rules in which an extra step function $\theta(|k_\ell|^2 - \lambda^2)$ is associated with each line. $i_\lambda(\mathcal{G})$ is defined to be the integrand of the graph \mathcal{G} .

3 The R operation

We now apply the simple momentum space decomposition which says that at every point in the space of loop momenta k some line ℓ has to carry the smallest momentum:

$$I_\lambda(\mathcal{G}) = \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk i_k(\mathcal{G}/\mathcal{H}(\mathcal{G}, \ell)) \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} I_k(\Theta).$$

For each henge all possible subdivergences of $I(\mathcal{G})$ must live within one of the “effective vertices” Θ , so it is most natural to define the \bar{R} operation, which subtracts all subdivergences, as

$$\bar{R}I_\lambda(\mathcal{G}) \equiv \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk i_k(\mathcal{G}/\mathcal{H}(\mathcal{G}, \ell)) \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} \bar{R}I_k(\Theta), \quad \bar{R}I_\lambda(\mathcal{G}) \equiv \bar{R}I_\lambda(\mathcal{G}) - K\bar{R}I_0(\mathcal{G}), \quad (1)$$

where R is the operation which subtracts all divergences.

3.1 The subtraction operation $-K$

The subtraction operator $-K$ removes the divergent part of $I(\mathcal{G})$. Various choices are possible

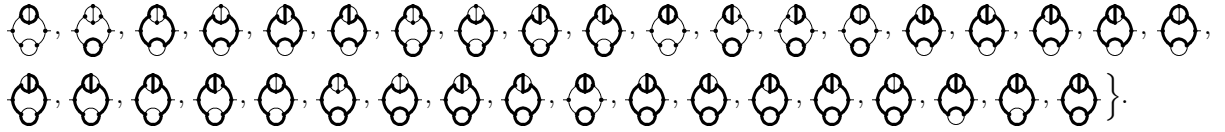
- For minimal subtraction $-KI(\mathcal{G})$ subtracts the pole terms in the Laurent expansion of $I(\mathcal{G})$ in the dimension D . In this case the BPH theorem states that these subtractions are local; i.e., polynomial in the external momenta p .
- K can be chosen to be the Taylor series subtraction operator $T^{\deg \mathcal{G}} I(\mathcal{G})$ with respect to the external momenta p , where $\deg \mathcal{G}$ is the overall (power counting) degree of divergence of \mathcal{G} . In this case the BPH theorem states that the subtracted Feynman integrals are convergent, i.e., they have a finite limit as the cutoff $\Lambda \rightarrow \infty$.

The subtraction operation commutes with differentiation: for minimal subtraction $[\partial, K] = 0$ trivially, whereas for Taylor series subtraction $\partial T^n = T^{n-1} \partial$ but, as we shall show in sections 4 and 5, $\deg \partial \mathcal{G} = \deg \mathcal{G} - 1$, so $[\partial, T^{\deg}] = 0$.

Strictly speaking we define $-K$ to replace the divergent part with a finite polynomial of degree $\deg \mathcal{G}$ in the external momenta. The finite part of a subtracted graph is specified unambiguously by some set of *renormalization conditions*, which fix the values of $I(p_0), \partial I(p_0), \dots, \partial^{\deg \mathcal{G}} I(p_0)$ at the *subtraction point* p_0 . As these renormalization conditions have no loop corrections this only affects tree-level diagrams, and leads us to the following conventions² for the graph v consisting of a single vertex: $RI_\lambda(v) = I_0(v)$, $\bar{R}I_\lambda(v) = 0$, and $-K\bar{R}I_0(v) = I_0(v)$.

3.2 Equivalence to Bogoliubov's Definition

A *spinney* [11] is a covering of a graph by a set of disjoint 1PI subgraphs. Single vertices are allowed as elements of spinneys:³ in other words, all the vertices of a graph are included in a spinney, but not necessarily all the edges. The *wood* $\mathcal{W}(\mathcal{G})$ is the set of all spinneys for a graph \mathcal{G} . Note that every henge is a spinney, but not vice versa. We shall use the notation $I_\lambda(\mathcal{G}/\mathcal{S} \star \prod_{\Theta \in \mathcal{S}} f(\Theta))$ to mean the Feynman integral for the graph \mathcal{G}/\mathcal{S} where all internal lines carry momentum larger than λ and the function $f(\Theta)$ is the Feynman rule for the “effective vertex” Θ . The *proper wood* $\bar{\mathcal{W}}(\mathcal{G})$ is just the wood with the spinney $\mathcal{S} = \mathcal{G}$ omitted. The following is an example of a wood from ϕ^3 theory: $\mathcal{W}(\text{triangle}) = \{ \text{triangle}, \text{triangle with one loop}, \text{triangle with two loops} \}$,



Bogoliubov's [3, 4] definition of the R operation is

$$\begin{aligned} \bar{R}_B I_\lambda(\mathcal{G}) &\equiv \sum_{\mathcal{S} \in \bar{\mathcal{W}}(\mathcal{G})} I_\lambda \left(\mathcal{G}/\mathcal{S} \star \prod_{\Gamma \in \mathcal{S}} -K\bar{R}_B I_0(\Gamma) \right), \\ R_B I_\lambda(\mathcal{G}) &\equiv \bar{R}_B I_\lambda(\mathcal{G}) - K\bar{R}_B I_0(\mathcal{G}) = \sum_{\mathcal{S} \in \mathcal{W}(\mathcal{G})} I_\lambda \left(\mathcal{G}/\mathcal{S} \star \prod_{\Gamma \in \mathcal{S}} -K\bar{R}_B I_0(\Gamma) \right), \end{aligned} \quad (2)$$

where we have made a generalization to allow a non-vanishing λ . We shall prove the equivalence of our definition of equation (1) with that of equation (2) by induction. For graphs with no loops the equivalence is trivial, and we assume that for graphs with fewer than L loops $RI_\lambda(\Gamma) = R_B I_\lambda(\Gamma)$. For an L loop graph \mathcal{G} equation (1) gives us

$$\begin{aligned} \bar{R}I_\lambda(\mathcal{G}) &= \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk i_k(\mathcal{G}/\mathcal{H}) \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} \sum_{\mathcal{S} \in \mathcal{W}(\Theta)} I_k \left(\Theta/\mathcal{S} \star \prod_{\Gamma \in \mathcal{S}} -K\bar{R}_B I_0(\Gamma) \right) \\ &= \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk i_k(\mathcal{G}/\mathcal{H}) \prod_{\mathcal{S} \in \mathcal{W}(\mathcal{H}(\mathcal{G}, \ell))} I_k \left(\mathcal{H}/\mathcal{S} \star \prod_{\Gamma \in \mathcal{S}} -K\bar{R}_B I_0(\Gamma) \right), \end{aligned} \quad (3)$$

²These are different from the usual conventions, but make the formalism tidier.

³This differs from the usual definition because of our conventions for single vertices.

where we have defined $\mathcal{W}(\mathcal{H})$ to be the set of all spinneys which lie within \mathcal{H} .⁴ It is clear that $\mathcal{S} \in \bar{\mathcal{W}}(\mathcal{G}) \Rightarrow \exists \ell : \mathcal{S} \in \mathcal{W}(\mathcal{H}(\mathcal{G}, \ell))$: just choose any line $\ell \notin \mathcal{S}$. It is also clear that $\mathcal{S} \in \mathcal{W}(\mathcal{H}(\mathcal{G}, \ell)) \Rightarrow \mathcal{S} \in \bar{\mathcal{W}}(\mathcal{G})$: just note that $\ell \notin \mathcal{H}(\mathcal{G}, \ell)$. Hence all of the subtractions in equation (3) correspond exactly to the spinneys in $\bar{\mathcal{W}}(\mathcal{G})$, and the subtractions corresponding to some such spinney are

$$\sum_{\ell \in \mathcal{G} : \mathcal{S} \in \mathcal{W}(\mathcal{H}(\mathcal{G}, \ell))} \int_{\lambda}^{\infty} dk i_k(\mathcal{G}/\mathcal{H}) I_k \left(\mathcal{H}/\mathcal{S} \star \prod_{\Gamma \in \mathcal{S}} -K \bar{R}_B I_0(\Gamma) \right) = I_{\lambda} \left(\mathcal{G}/\mathcal{S} \star \prod_{\Gamma \in \mathcal{S}} -K \bar{R}_B I_0(\Gamma) \right),$$

because the set of lines ℓ for which $\mathcal{S} \in \mathcal{W}(\mathcal{H}(\mathcal{G}, \ell))$ is precisely \mathcal{G}/\mathcal{S} . Therefore we have shown that $\bar{R}I_{\lambda}(\mathcal{G}) = \bar{R}_B I_{\lambda}(\mathcal{G})$.

The definition of the R-operation can be made even more explicit and less recursive using Zimmermann's [17] forest notation: however it is easier to construct proofs and write computer programs to automate renormalization using recursive definitions.

In Bogoliubov's form it is manifest that $[\partial, R] = 0$, because (i) $[\partial, K] = 0$, and (ii) the definition of R is purely graphical, and the graphical structure is not changed by differentiation.

3.3 Equivalence to Counterterms

We shall show that the subtractions made by the R operation are equivalent to the addition of counterterms to the action. As this is a purely combinatorial proof it is convenient to use the generating functional $Z(J) = \langle e^{-S(\phi) + J\phi} \rangle = \exp[-S_I(\frac{\delta}{\delta J})] e^{\frac{1}{2}J\Delta J}$, where $S(\phi) = \frac{1}{2}\phi\Delta^{-1}\phi + S_I(\phi)$. Perturbation theory may be viewed as an expansion in the number of vertices in a graph,

$$Z(J) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} [S_I(\frac{\delta}{\delta J})]^n e^{\frac{1}{2}J\Delta J} = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\mathcal{G}_n} I(\mathcal{G}_n(J)) e^{\frac{1}{2}J\Delta J};$$

where the last sum is over all graphs \mathcal{G}_n containing exactly n vertices and which have J attached to their external legs. We define the *renormalized generating functional* as

$$RZ(J) \equiv \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\mathcal{G}_n} R I(\mathcal{G}_n) e^{\frac{1}{2}J\Delta J} = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\mathcal{G}_n} \sum_{\mathcal{S} \in \mathcal{W}(\mathcal{G}_n)} I\left(\mathcal{G}/\mathcal{S} \star \prod_{\Gamma \in \mathcal{S}} -K \bar{R} I(\Gamma)\right) e^{\frac{1}{2}J\Delta J}.$$

Using the identity

$$\sum_{\mathcal{G}_n} \sum_{\mathcal{S} \in \mathcal{W}(\mathcal{G}_n)} \prod_{\Gamma \in \mathcal{S}} -K \bar{R} I(\Gamma) = \sum_{\substack{r_0, \dots, r_n \\ r_0 + \dots + r_n = n}} \frac{n!}{\prod_{j=0}^n j!^{r_j} r_j!} \prod_{j=0}^n \left[\sum_{\mathcal{G}_j} -K \bar{R} I(\mathcal{G}_j) \right]^{r_j}$$

where the last sum is over all graphs \mathcal{G}_j with exactly j vertices, we obtain

$$\begin{aligned} RZ(J) &= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\substack{r_0, \dots, r_n \\ r_0 + \dots + r_n = n}} \frac{n!}{\prod_{j=0}^n j!^{r_j} r_j!} \prod_{j=0}^n \left[\sum_{\mathcal{G}_j} -K \bar{R} I(\mathcal{G}_j(\frac{\delta}{\delta J})) \right]^{r_j} e^{\frac{1}{2}J\Delta J} \\ &= \prod_{j=0}^{\infty} \sum_{r_j=0}^{\infty} \frac{1}{r_j!} \left[\frac{1}{j!} \sum_{\mathcal{G}_j} K \bar{R} I(\mathcal{G}_j(\frac{\delta}{\delta J})) \right]^{r_j} e^{\frac{1}{2}J\Delta J} = \prod_{j=0}^{\infty} \exp \left[\frac{1}{j!} \sum_{\mathcal{G}_j} K \bar{R} I(\mathcal{G}_j(\frac{\delta}{\delta J})) \right] e^{\frac{1}{2}J\Delta J} \\ &= \exp \left[K \bar{R} e^{S_I(\frac{\delta}{\delta J})} \right] e^{\frac{1}{2}J\Delta J}. \end{aligned}$$

We have thus shown that $RZ(J) = \langle e^{-S_B(\phi) + J\phi} \rangle$, with the *bare action*

$$S_B(\phi) = \frac{1}{2}\phi\Delta^{-1}\phi - K \bar{R} e^{S_I(\phi)}. \quad (4)$$

⁴To be precise, $\mathcal{W}(\mathcal{H})$ is the set of spinneys which are sets of disjoint 1PI subgraphs which are either single vertices or are subgraphs of one of the 1PI subgraphs of \mathcal{G} in the hinge \mathcal{H} .

Observe that there is no simple one to one correspondence between countergraphs and subtractions, but that the combinatorial factors arrange themselves correctly. The counterterms are monomials in the bare action, and we draw the fields ϕ or functional derivatives $\frac{\delta}{\delta J}$ by open circles at the end of the amputated external legs; such graphs are symmetric under interchange of their external legs. The appropriate combinatorial factors must be used for each graph. Some of the counterterms in ϕ^3 theory in D dimensions are

$$\begin{aligned} \bullet &= \frac{1}{2} \bigcirc + \frac{1}{4} \bigcirc \! \! \bigcirc + O(\hbar^3) & \bullet \! \! \bullet &= \bigcirc \! \! \bigcirc + \frac{3}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc + \frac{3}{4} \bigcirc \! \! \bigcirc \! \! \bigcirc \! \! \bigcirc + O(\hbar^3) \\ \bullet \! \! \bullet &= \frac{1}{2} \bigcirc \! \! \bigcirc + \frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc + \bigcirc \! \! \bigcirc \! \! \bigcirc + O(\hbar^3) & \bullet \! \! \bullet \! \! \bullet &= 3 \bigcirc \! \! \bigcirc \! \! \bigcirc + O(\hbar^2) \end{aligned}$$

The countergraphs built using these counterterms correspond to subtractions in the following way:

	$\frac{1}{2} \bigcirc \! \! \bigcirc$	$\frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc$	$\frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc$	$\bigcirc \! \! \bigcirc \! \! \bigcirc$	\bullet
$\frac{1}{2} \bigcirc \! \! \bigcirc$	$\frac{1}{2} \bigcirc \! \! \bigcirc$	$\frac{1}{2} \bigcirc \! \! \bigcirc$	$\frac{1}{2} \bigcirc \! \! \bigcirc$		$\frac{1}{2} \bigcirc \! \! \bigcirc$
$\frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc$	$\frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc + \frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc$			$\frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc$	$\frac{1}{2} \bigcirc \! \! \bigcirc \! \! \bigcirc$

4 Bounding Inequalities

A condition for applicability of our proof of the BPH theorem is that we require certain bounds to hold at tree level. All vertices and propagators Γ satisfy $|I_\lambda(\Gamma)| \leq c \cdot \chi(\lambda)^{\deg \Gamma}$, where c is a constant, and the *overall degree of divergence* $\deg \Gamma$ is a number which will be used for power counting. The monotonically increasing bounding function χ must satisfy

$$\int_\lambda^\infty dk \chi(k)^\nu \leq c \cdot \chi(\lambda)^{\nu+1} \quad (\nu + 1 < 0), \quad \int_0^\lambda dk \chi(k)^\nu \leq c \cdot \chi(\lambda)^{\nu+1+0} \quad (\nu + 1 \geq 0). \quad (5)$$

Differentiation with respect to external momenta must lower the degree of divergence, $\deg(\partial \mathcal{G}) = \deg \mathcal{G} - 1$. This means that we also require that all derivatives of vertices and propagators must satisfy the bounds

$$|\partial^n I_\lambda(\Gamma)| \leq c \cdot \chi(\lambda)^{\deg \Gamma - n}. \quad (6)$$

The simple choice $\chi(k) \equiv \max(m, k)$ suffices for our proof of the BPH theorem: a simple generalization is needed to prove the decoupling theorem. For the lattice propagator $\Delta(k) = \left[m^2 + \frac{4}{a^2} \sum_{\mu=1}^D \sin^2 \frac{1}{2} k_\mu a \right]^{-1}$ the inequality $2|x|/\pi \leq |\sin x| \leq 1$ may be used to show that (6) holds.

For “sharp cutoffs” like a lattice regulator for which each propagator has a factor $\theta(k_\ell - \pi/a)$ there are also “surface terms” which arise in derivatives of graphs. The generalization of equation (6) to include these terms is straightforward.

5 Proof of the BPH Theorem

Our proof uses the induction hypothesis that $|RI_\lambda(\mathcal{G})| \leq c \cdot \chi(\lambda)^{\deg \mathcal{G} + 0}$ for all graphs with fewer than L loops. For $L = 0$ this follows trivially from the bounding inequalities of the previous section. To show that it continues to hold for L -loop graphs we consider two cases.

5.1 Overall convergent diagrams with L loops

From the definition (1) of \bar{R} we have

$$|\bar{R}I_\lambda(\mathcal{G})| \leq \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk |i_k(\mathcal{G}/\mathcal{H})| \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} |RI_k(\Theta)|.$$

Using the induction hypothesis for the subgraphs Θ and the tree level bounds (6) for $i_k(\mathcal{G}/\mathcal{H})$ we get

$$|\bar{R}I_\lambda(\mathcal{G})| \leq c \cdot \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk \chi(k)^{\deg(\mathcal{G}/\mathcal{H})-1} \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} \chi(k)^{\deg \Theta + 0} = c \cdot \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk \chi(k)^{\deg \mathcal{G} - 1 + 0},$$

and upon integrating the bounding function using (5) we find $|\bar{R}I_\lambda(\mathcal{G})| \leq c \cdot \chi(\lambda)^{\deg \mathcal{G} + 0}$ for $\deg \mathcal{G} < 0$. This establishes the induction hypothesis, since $RI_\lambda(\mathcal{G}) = \bar{R}I_\lambda(\mathcal{G})$ in this case.

5.2 Overall divergent diagrams with L loops

Taylor's theorem for the function $\bar{R}I_0(p)$ gives

$$\bar{R}I_0(p) = T^{\deg \mathcal{G}} \bar{R}I_0(p) + \int_{p_0}^p dp_1 \dots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1} \partial^{\deg \mathcal{G}+1} \bar{R}I_0(p_{\deg \mathcal{G}+1}),$$

and since \bar{R} and ∂ commute

$$\bar{R}I_0(p) = T^{\deg \mathcal{G}} \bar{R}I_0(p) + \int_{p_0}^p dp_1 \dots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1} \bar{R} \partial^{\deg \mathcal{G}+1} I_0(p_{\deg \mathcal{G}+1}).$$

The (sum of) graphs $\partial^{\deg \mathcal{G}+1} \mathcal{G}$ are overall convergent, so the integrand is finite, and as the integral is over a compact region any divergences must be in the polynomial part. The \bar{R} operation removes this polynomial $T^{\deg \mathcal{G}} \bar{R}I_0(p)$, so $|\bar{R}I_0(p)| \leq \int_{p_0}^p dp_1 \dots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1} |R \partial^{\deg \mathcal{G}+1} I_0(p_{\deg \mathcal{G}+1})|$. Using the inductive bound already established for the overall convergent integrand $|RI_0(p)| \leq \int_{p_0}^p dp_1 \dots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1} c \cdot \chi(0)^{-1+0} \leq c \cdot \chi(0)^{\deg \mathcal{G}+0}$ we prove that $RI_0(\mathcal{G})$ is made finite by local subtractions, but we still need to establish the induction hypothesis. In the definition of $\bar{R}I_0(\mathcal{G})$ we may split the integration region $\int_0^\infty dk = \int_0^\lambda dk + \int_\lambda^\infty dk$, hence

$$RI_0(\mathcal{G}) = RI_\lambda(\mathcal{G}) + \sum_{\ell \in \mathcal{G}} \int_0^\lambda dk i_k(\mathcal{G}/\mathcal{H}) \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} RI_k(\Theta).$$

All that is left to do is to bound the integral over the “infrared region” using essentially the same technique as for the overall convergent case above

$$\begin{aligned} |RI_\lambda(\mathcal{G})| &\leq c \cdot \chi(0)^{\deg \mathcal{G}+0} + c \cdot \sum_{\ell \in \mathcal{G}} \int_0^\lambda dk \chi(k)^{\deg(\mathcal{G}/\mathcal{H})-1} \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} \chi(k)^{\deg \Theta + 0} \\ &\leq c \cdot \chi(0)^{\deg \mathcal{G}+0} + c \cdot \sum_{\ell \in \mathcal{G}} \int_0^\lambda dk \chi(k)^{\deg \mathcal{G}-1+0} \leq c \cdot \chi(\lambda)^{\deg \mathcal{G}+0} \quad (\deg \mathcal{G} \geq 0). \end{aligned}$$

6 Power Counting

Consider a connected Feynman diagram \mathcal{G} in a D dimensional field theory with an arbitrary polynomial action. Let it have I_a lines of type a , V_b vertices of type b , and E_a external legs of type a . Let n_{ab} be the number of lines of type a which are attached to vertex b , d'_b be the degree of this vertex, and d_a be the degree of lines of type a . Every line has to end on an appropriate vertex, so $\sum_b n_{ab} V_b = E_a + 2I_a$ ($\forall a$). We require exactly $V - 1$ lines to connect V vertices into a tree; every extra line produces a loop: hence $L = I - V + 1 = \sum_a I_a - \sum_b V_b + 1$. The overall degree of the graph can be obtained by counting, $\deg \mathcal{G} = LD + \sum_b V_b d'_b + \sum_a I_a d_a$. Eliminating L and I_a from these equations we obtain

$$\deg \mathcal{G} = \sum_b V_b \left[\frac{1}{2} \sum_a \{n_{ab}(d_a + D)\} + d'_b - D \right] - \frac{1}{2} \sum_a E_a (d_a + D) + D.$$

The *dimension* of the monomial \mathcal{V}_b in the action⁵ corresponding to the vertex of type b may be defined to be $\dim \mathcal{V}_b \equiv \sum_a n_{ab} \dim(\phi_a) + d'_b - D$, where the dimension of the field ϕ_a is defined such that

⁵This differs by D from the dimension of the corresponding monomial in the Lagrangian density.

the dimension of its kinetic term in the action vanishes; that is, $\dim \phi_a \equiv \frac{1}{2}(d_a + D)$. This gives $\dim \mathcal{V}_b = \frac{1}{2} \sum_a n_{ab}(d_a + D) + d'_b - D$. We thus obtain

$$\deg \mathcal{G} = \sum_b V_b \dim \mathcal{V}_b - \sum_a E_a \dim \phi_a + D. \quad (7)$$

A theory is *superrenormalizable*, that is has only a finite number of overall divergent graphs, if the coefficients of V_b are negative: $\dim \mathcal{V}_b < 0$ ($\forall b$). The theory is *renormalizable*, that is only a finite number of Green's functions are overall divergent, if none of the coefficients of V_b are positive, $\dim \mathcal{V}_b \leq 0$ ($\forall b$), and all the coefficients of E_a are positive, $\dim \phi_a > 0$ ($\forall a$). In general, all local monomials of dimension ≤ 0 will be required as counterterms.

7 Some Applications

7.1 Operator Insertions

Let $\Omega(\phi)$ be an operator which is local and polynomial in the field ϕ . Add a source term for Ω into the action, $Z(J, J') \equiv \langle e^{-S(\phi) + J\phi + J'\Omega(\phi)} \rangle$. The BPH theorem tells us that this theory can be renormalized by adding local counterterms of the form given by equation (4), $\Delta S(\phi, J') = -S_I(\phi) + J'\Omega(\phi) - \text{K}\bar{\text{R}} \exp[S_I(\phi) - J'\Omega(\phi)]$. Expanding in powers of J' gives $\Delta S(\phi, 0) + J' \text{K}\bar{\text{R}} [e^{S_I(\phi)} \Omega(\phi)] + J'\Omega(\phi) + O(J'^2)$. We may associate the counterterms linear in J' with the operator to define a renormalized operator $N(\Omega) \equiv -\text{K}\bar{\text{R}} [e^{S_I(\phi)} \Omega(\phi)]$. Power counting (7) tells us that $\deg \mathcal{G} = V_\Omega \dim \Omega + \sum_b V_b \dim \mathcal{V}_b - \sum_a E_a \dim \phi_a + D$, where V_Ω are the number of Ω vertices in \mathcal{G} . As we are interested in a single insertion of Ω we only care about counterterms linear in J' , and these get contributions only from diagrams \mathcal{G} with $V_\Omega = 1$. Thus $\deg \mathcal{G} \leq \dim \Omega + D$, for a renormalizable theory, which means that we only get counterterms of dimension $\leq \dim \Omega$. Analogous arguments easily establish the operator product expansion.

7.2 Oversubtraction

If one subtracts more than $\deg \mathcal{G} + 1$ terms in the Taylor expansion in the external momenta from a graph then the graph does not become any more convergent (in the sense of lowering the exponent in the inductive bound of section 5), but the dependence on the cutoff parameter is reduced. This result of Zimmermann's [17] is central to Symanzik's improvement programme on the lattice [18]. Our methods may be used to prove Zimmermann's theorem by applying the arguments of section 5 to the derivatives of $I(\mathcal{G})$ with respect to the cutoff $a = 1/\Lambda$.

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References

- [1] F. J. Dyson *Phys. Rev.* **75** (1949) 486.
- [2] F. J. Dyson *Phys. Rev.* **75** (1949) 1736.
- [3] N. N. Bogoliubov and O. S. Parasiuk *Izv. Akad. Nauk SSSR* **20** (1956) 585.
- [4] N. N. Bogoliubov and O. S. Parasiuk *Acta. Math.* **97** (1957) 227.

- [5] E. C. G. Stückelberg and T. A. Green *Helv. Phys. Acta* **24** (1951) 153.
- [6] K. Hepp *Commun. Math. Phys.* **2** (1966) 301.
- [7] A. A. Vladimirov, “Method for computing renormalization group functions in dimensional renormalization scheme,” *Theor. Math. Phys.* **43** (1980) 417 JINR–E2–12248.
- [8] O. V. Tarasov and A. A. Vladimirov, “Three loop calculations in nonabelian gauge theories,” tech. rep., JINR, 1980. E2–80–483.
- [9] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, “New approach to evaluation of multiloop Feynman integrals: the Gegenbauer polynomial x space technique,” *Nucl. Phys.* **B174** (1980) 345–377.
- [10] S. A. Anikin, M. K. Polivanov, and O. I. Zavialov, “A simple proof of the Bogolyubov–Parasiuk theorem,” *Theor. Math. Phys.* **17** (1973) 1082.
- [11] W. E. Caswell and A. D. Kennedy, “A simple approach to renormalization theory,” *Phys. Rev.* **D25** (1982) 392.
- [12] Y. Hahn and W. Zimmermann *Commun. Math. Phys.* **10** (1968) 330.
- [13] W. E. Caswell and A. D. Kennedy, “The asymptotic behaviour of Feynman integrals,” *Phys. Rev.* **D26** (1983) 3073.
- [14] T. Reisz, “Renormalization of Feynman integrals on the lattice,” *Commun. Math. Phys.* **117** (1988) 79.
- [15] T. Reisz, “A power counting theorem for Feynman integrals on the lattice,” *Commun. Math. Phys.* **116** (1988) 81.
- [16] W. Zimmermann *Commun. Math. Phys.* **11** (1968) 1.
- [17] W. Zimmermann in *Lectures on Elementary Particles and Quantum Field Theory* (S. Deser, M. Grisaru, and H. Pendleton, eds.), (Cambridge, Mass.), MIT, 1970.
- [18] K. Symanzik, “Continuum limit and improved action in lattice theories. 1. Principles and ϕ^4 theory,” *Nucl. Phys.* **B226** (1983) 187.